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SIMULTANEOUS CONFIDENCE BANDS FOR AUTOREGRESSIVE SPECTRA.(U)

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SIMULTANEOUS CONFIDENCE BANDS
FOR AUTOREGRESSIVE SPECTRA

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1. INTRODUCTION

Autoregressive spectral estimation has become an important method of spectral estimation in recent years (see Akaike (1969), Parzen (1974), Ulrych and Bishop (1975), and Beamish and Priestley (1981), for example) despite 1) a continuing discussion of the problems of order determination and 2) a lack of easily applied procedures for determining confidence intervals or bands on the function being estimated.

In this paper we derive asymptotic $100(1-\alpha)\%$ simultaneous confidence bands for an autoregressive spectral density assuming one has data from a finite order autoregressive process with known order or conditional upon having correctly determined the order if it is unknown.

The bands are derived in section 2 and their implementation described on simulated data in section 3.

2. SIMULTANEOUS CONFIDENCE BANDS

The basic property of autoregressive processes that we shall use is that the reciprocal of the autoregressive spectral estimator is a linear combination of a finite number of asymptotically normal random variables whose asymptotic covariance matrix is easily consistently estimated. Thus one can use a Scheffé (1953) type projection argument to determine asymptotic confidence bands on the reciprocal of the autoregressive spectral density (and thus on the density itself) at all frequencies.

Let Y be an autoregressive process of order p with coefficients $\alpha(1), \dots, \alpha(p)$ and noise variance σ^2 , i.e.

$$\sum_{j=0}^p \alpha(j) Y(t-j) = \varepsilon(t) \quad , \quad t=0, \underline{+1}, \underline{+2}, \dots$$

where $\alpha(0) = 1$, ε is a white noise series of uncorrelated zero mean random variables having common variance σ^2 , and the zeros of the complex valued polynomial $g(z) = \sum_{j=0}^p \alpha(j) z^j$ are all greater than one in modulus. Then we write $Y \sim \text{AR}(p, \underline{\alpha}, \sigma^2)$ and note that the covariance function $R(v) = E(Y(t)Y(t+v))$, $v=0, \underline{+1}, \dots$, the spectral density function $f(\omega) = \frac{1}{2\pi} \sum_{v=-\infty}^{\infty} R(v) e^{-iv\omega}$, $\omega \in [-\pi, \pi]$, and the parameters $\underline{\alpha}$ and σ^2 are related by

$$f(\omega) = \frac{\sigma^2}{2\pi} \frac{1}{|g(e^{i\omega})|^2} \quad , \quad \omega \in [-\pi, \pi], \quad (2.1)$$

$$\sum_{j=0}^p \alpha(j) R(j-v) = \delta_v \sigma^2 \quad , \quad v \geq 0, \quad (2.2)$$

where δ_v is the Kronecker delta. If on the other hand, Y is a moving average process of order q and parameters $\beta(1), \dots, \beta(q)$, and σ^2 , i.e

$$Y(t) = \sum_{k=0}^q \beta(k) \varepsilon(t-k) \quad , \quad t=0, \underline{+1}, \dots,$$

where $\beta(0) = 1$, we have, with $h(z) = \sum_{k=0}^q \beta(k) z^k$,

$$R(v) = \begin{cases} \sigma^2 \sum_{k=0}^{q-|v|} \beta(k) \beta(k+|v|) & , \quad |v| \leq q \\ 0 & , \quad |v| > q \end{cases} \quad ,$$

$$\begin{aligned}
 f(\omega) &= \frac{1}{2\pi} \sum_{v=-q}^q R(v) e^{-iv\omega} \\
 &= \frac{1}{2\pi} [R(0) + 2 \sum_{v=1}^q R(v) \cos v\omega] \\
 &= \frac{\sigma^2}{2\pi} |h(e^{i\omega})|^2, \quad \omega \in [-\pi, \pi].
 \end{aligned}$$



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Thus, if $Y \sim \text{AR}(p, \alpha, \sigma^2)$ we have $f(\omega) > 0$ for all ω and

$$h(\omega) = \frac{1}{f(\omega)} = \frac{4\pi^2/\sigma^2}{2\pi} |g(e^{i\omega})|^2 = \underline{x}^T(\omega) \underline{\gamma}$$

where $\underline{x}^T(\omega) = (\frac{1}{2\pi}, \frac{1}{\pi} \cos \omega, \dots, \frac{1}{\pi} \cos p\omega)$, and $\underline{\gamma}^T = (\gamma(0), \gamma(1), \dots, \gamma(p))$, with

$$\gamma(v) = \frac{4\pi^2}{\sigma^2} \sum_{j=0}^{p-v} \alpha(j) \alpha(j+v), \quad v=0, \dots, p.$$

Thus the reciprocal of f is a linear combination of the parameters $\gamma(0), \gamma(1), \dots, \gamma(p)$.

Let $Y \sim \text{AR}(p, \alpha, \sigma^2)$ and let $Y(1), \dots, Y(n)$ be a sample realization from Y . Let $\hat{\underline{\theta}}^T = (\hat{\underline{\alpha}}^T, \hat{\sigma}^2) = (\hat{\alpha}(1), \dots, \hat{\alpha}(p), \hat{\sigma}^2)$ be estimators of the parameters $\underline{\theta}^T = (\underline{\alpha}^T, \sigma^2) = (\alpha(1), \dots, \alpha(p), \sigma^2)$ that have the same asymptotic distribution as the maximum likelihood estimators of $\underline{\theta}$, i.e (see Parzen (1961))

$$\sqrt{n} (\hat{\underline{\theta}} - \underline{\theta}) \xrightarrow{D} N_{p+1}(0, \underline{f}(\underline{\theta})) \quad (2.3)$$

where

$$\hat{\theta}(\theta) = \begin{bmatrix} \sigma^2 \Gamma_p^{-1} & 0 \\ 0^T & 2\sigma^4 \end{bmatrix},$$

and Γ_p is the $(p \times p)$ Toeplitz covariance matrix of $Y(1), \dots, Y(p)$. Estimators having this asymptotic distribution include

(1) The Yule-Walker estimators (YWE), i.e the solution to the Yule-Walker equations (2.2) with

$$\hat{R}(v) = \frac{1}{T} \sum_{t=1}^{T-v} Y(t)Y(t+v), \quad v=0, \dots, p$$

replacing $R(v)$.

(2) The Least Squares estimators (LSE), i.e the values $\hat{\alpha}(1), \dots, \hat{\alpha}(p)$ minimizing

$$S(\alpha) = \sum_{t=p+1}^T \{Y(t) - \alpha(1)Y(t-1) - \dots - \alpha(p)Y(t-p)\}^2,$$

and $\hat{\sigma}^2 = S(\hat{\alpha})/(n-p)$.

3) The Burg estimators (BGE), (see Ulrych and Bishop (1975) for a description of these estimators).

Then the autoregressive spectral estimator \hat{f} of f consists of evaluating (2.1) with $\hat{\theta}$ replacing θ . If the order p is unknown a priori there are several procedures (see Hannan and Quinn (1979), for example) for finding a value \hat{p} from the data which as $n \rightarrow \infty$ has probability one of determining the correct order p .

Then the confidence bands are given by the following Theorem:

Theorem 2.1 Let $Y(1), \dots, Y(n)$ be a sample realization from a Gaussian $AR(p, \alpha, \sigma^2)$ time series and let $\hat{\alpha}(1), \dots, \hat{\alpha}(p)$, and $\hat{\sigma}^2$ be estimators satisfying (2.3). Let

$$\hat{\gamma}(v) = \frac{4\pi^2}{\hat{\sigma}^2} \sum_{j=0}^{p-v} \hat{\alpha}(j) \hat{\alpha}(j+v) \quad , \quad v=0, \dots, p$$

and $\hat{h}(\omega) = \frac{1}{\hat{f}(\omega)} = \frac{1}{2\pi} [\hat{\gamma}(0) + 2 \sum_{v=1}^p \hat{\gamma}(v) \cos v\omega]$. Let $\underline{\beta}^T = (\underline{\alpha}^T, 1/\sigma^2)$,

$\hat{\underline{\beta}}^T = (\hat{\underline{\alpha}}^T, 1/\hat{\sigma}^2)$, and $B(\underline{\beta})$ be the $(p+1) \times (p+1)$ matrix having (ℓ, m) element

$$B_{\ell m}(\underline{\beta}) = \begin{cases} \sigma^2 \gamma(\ell-1) & , \quad m=p+1 \\ \frac{4\pi^2}{\sigma^2} \delta_1(\ell, m) + \frac{4\pi^2}{\sigma^2} \delta_2(\ell, m) & , \quad m=1, \dots, p \end{cases}$$

$\ell=1, \dots, p+1$, where $\delta_1(\ell, m) = \alpha(m+(\ell-1))$ if $m \leq p-(\ell-1)$ and zero otherwise, while $\delta_2(\ell, m) = \alpha(m-(\ell-1))$ if $m \geq \ell-1$ and zero otherwise. Then:

a) $\sqrt{n} (\hat{\underline{\gamma}} - \underline{\gamma}) \xrightarrow{D} N_{p+1} (0, \underline{\mathfrak{I}}(\underline{\gamma}))$ where $\underline{\mathfrak{I}}(\underline{\gamma}) = B(\underline{\beta}) \underline{\mathfrak{I}}(\underline{\beta}) B(\underline{\beta})^T$, and

$$\underline{\mathfrak{I}}(\underline{\beta}) = \begin{bmatrix} \sigma^2 \Gamma_p^{-1} & 0 \\ 0^T & 2/\sigma^4 \end{bmatrix}$$

b) Asymptotically, as $n \rightarrow \infty$, the probability is at least $1-\alpha$ that simultaneously for all $\omega \in [0, \pi]$,

$$\frac{1}{\hat{h}(\omega) + S(\omega)} \leq f(\omega) \leq \frac{1}{\hat{h}(\omega) - S(\omega)}$$

where if $\hat{h}(\omega) - S(\omega) \leq 0$ we use infinity as the upper limit, and

$$S^2(\omega) = \frac{\chi^2_{\alpha, p+1}}{n} x^T(\omega) \dagger(\hat{\gamma}) x(\omega),$$

with $\chi^2_{\alpha, p+1}$ being the upper α critical value of a Chi square distribution having $p+1$ degrees of freedom.

Proof To prove (a) we note that from (2.3) and the definition of β we have $\sqrt{n} (\hat{\beta} - \beta) \xrightarrow{D} N_{p+1}(0, \dagger(\beta))$. Then defining $c = 1/\sigma^2$ and $\hat{c} = 1/\hat{\sigma}^2$, we have for $v=0, \dots, p$,

$$\begin{aligned} \sqrt{n} [\hat{\gamma}(v) - \gamma(v)] &= 4\pi^2 \sum_{j=0}^{p-v} \sqrt{n} \{ \hat{c} \hat{\alpha}(j) \hat{\alpha}(j+v) - c \alpha(j) \alpha(j+v) \} \\ &= 4\pi^2 \sum_{j=0}^{p-v} \sqrt{n} \{ \hat{c} \hat{\alpha}(j) \hat{\alpha}(j+v) - \hat{c} \alpha(j) \hat{\alpha}(j+v) + \hat{c} \alpha(j) \hat{\alpha}(j+v) \\ &\quad - c \alpha(j) \hat{\alpha}(j+v) + c \alpha(j) \hat{\alpha}(j+v) - c \alpha(j) \alpha(j+v) \} \\ &= 4\pi^2 \sum_{j=0}^{p-v} \sqrt{n} \{ \hat{c} \hat{\alpha}(j+v) [\hat{\alpha}(j) - \alpha(j)] + \alpha(j) \hat{\alpha}(j+v) (\hat{c} - c) \\ &\quad + c \alpha(j) [\hat{\alpha}(j+v) - \alpha(j+v)] \} \\ &\sim 4\pi^2 \sum_{j=0}^{p-v} \{ c \alpha(j+v) \sqrt{n} [\hat{\alpha}(j) - \alpha(j)] + \alpha(j) \alpha(j+v) \sqrt{n} (\hat{c} - c) \\ &\quad + c \alpha(j) \sqrt{n} [\hat{\alpha}(j+v) - \alpha(j+v)] \} \\ &= 4\pi^2 \sum_{j=0}^{p-v} c \alpha(j+v) \sqrt{n} [\hat{\alpha}(j) - \alpha(j)] \\ &\quad + 4\pi^2 \sqrt{n} (\hat{c} - c) \sum_{j=0}^{p-v} \alpha(j) \alpha(j+v) + 4\pi^2 \sum_{j=v}^p c \alpha(j-v) \sqrt{n} [\hat{\alpha}(j) - \alpha(j)], \end{aligned}$$

where we write $X_n \approx Y_n$ to indicate that the sequences $\{X_n\}$, $\{Y_n\}$ of random variables converge in distribution to the same random variables, and we have used Slutsky's Theorem (Rao (1973), p. 122) repeatedly.

Thus we have shown that $\sqrt{n} (\hat{\gamma} - \gamma) \approx \sqrt{n} B(\beta)(\hat{\beta} - \beta)$ and (a) follows.

From (a) then we have

$$n(\hat{\gamma} - \gamma)^T \hat{I}^{-1}(\gamma) (\hat{\gamma} - \gamma) \xrightarrow{D} \chi_{p+1}^2$$

and since the elements of $\hat{I}(\gamma)$ are continuous functions of the elements of γ we have (Rao (1973), p. 388)

$$n(\hat{\gamma} - \gamma)^T \hat{I}^{-1}(\hat{\gamma}) (\hat{\gamma} - \gamma) \xrightarrow{D} \chi_{p+1}^2$$

Thus the probability is (asymptotically) $1 - \alpha$ that the true parameter γ lies inside the ellipsoid defined as the set of vectors γ satisfying

$$(\hat{\gamma} - \gamma)^T M (\hat{\gamma} - \gamma) \leq 1,$$

where $M = n \hat{I}^{-1}(\hat{\gamma}) / \chi_{\alpha, p+1}^2$ and $\chi_{\alpha, p+1}^2$ is the upper α critical value of a χ_{p+1}^2 distribution. But (Scheffe (1959), p. 407, et. sec.), γ is in this ellipsoid if and only if

$$|x^T (\hat{\gamma} - \gamma)| \leq [x^T M^{-1} x]^{\frac{1}{2}}$$

for all $(p+1)$ dimensional vectors x and thus in particular only if (for vectors $x(\omega)$)

$$|\hat{h}(\omega) - h(\omega)| \leq S(\omega), \quad [0, \pi],$$

giving a $100(1-\alpha)\%$ simultaneous confidence band for $h(\omega) = 1/f(\omega)$ as

$$\hat{h}(\omega) - S(\omega) \leq h(\omega) \leq \hat{h}(\omega) + S(\omega).$$

Then the reciprocal of this gives a simultaneous confidence band for f , where if $\hat{h}(\omega) - S(\omega) < 0$ we use infinity as the upper limit on $f(\omega)$ which we can do without decreasing the probability content of the bands.

3. EXAMPLES

To illustrate the method of section 2 we calculated the autoregressive spectral estimator and confidence bands for 10 sample realizations of length 200 for each of three AR processes (see Beamish and Priestley (1981)); i.e. AR(2, -.4, -.45, 1), AR(5, 1.7, 2.4, 1.634, .872, .168, 1), and AR(4, -2.7607, 3.8106, -2.6535, .9238, 1). These processes have characteristic polynomial zeros whose moduli are given by (1.11, 2.00), (1.12, 1.12, 1.19, 1.19, 3.33), and (1.01998, 1.01998, 1.019798, 1.019798) respectively, and have variances 2.66, 30.59, and 761.3. Further, the ratio of the maximum to the minimum values of the spectral densities for the three models are approximately 111.5, 41,800, and 6×10^6 . Thus the performance of our method on these models should be representative of its performance on a wide class of AR models. In Figures A-C we display on a log scale typical point estimators and 95% confidence bands for the spectral density of the AR(2), AR(5), and AR(4) models respectively using the Burg estimators. The true

spectral density (the curve with the x's) is superimposed on the graphs. Each curve consists of connecting values of the various functions at the $|u|$ equally spaced points between 0 and π (labelled on the graph 0 to $1/2$ cycles per sampling interval).

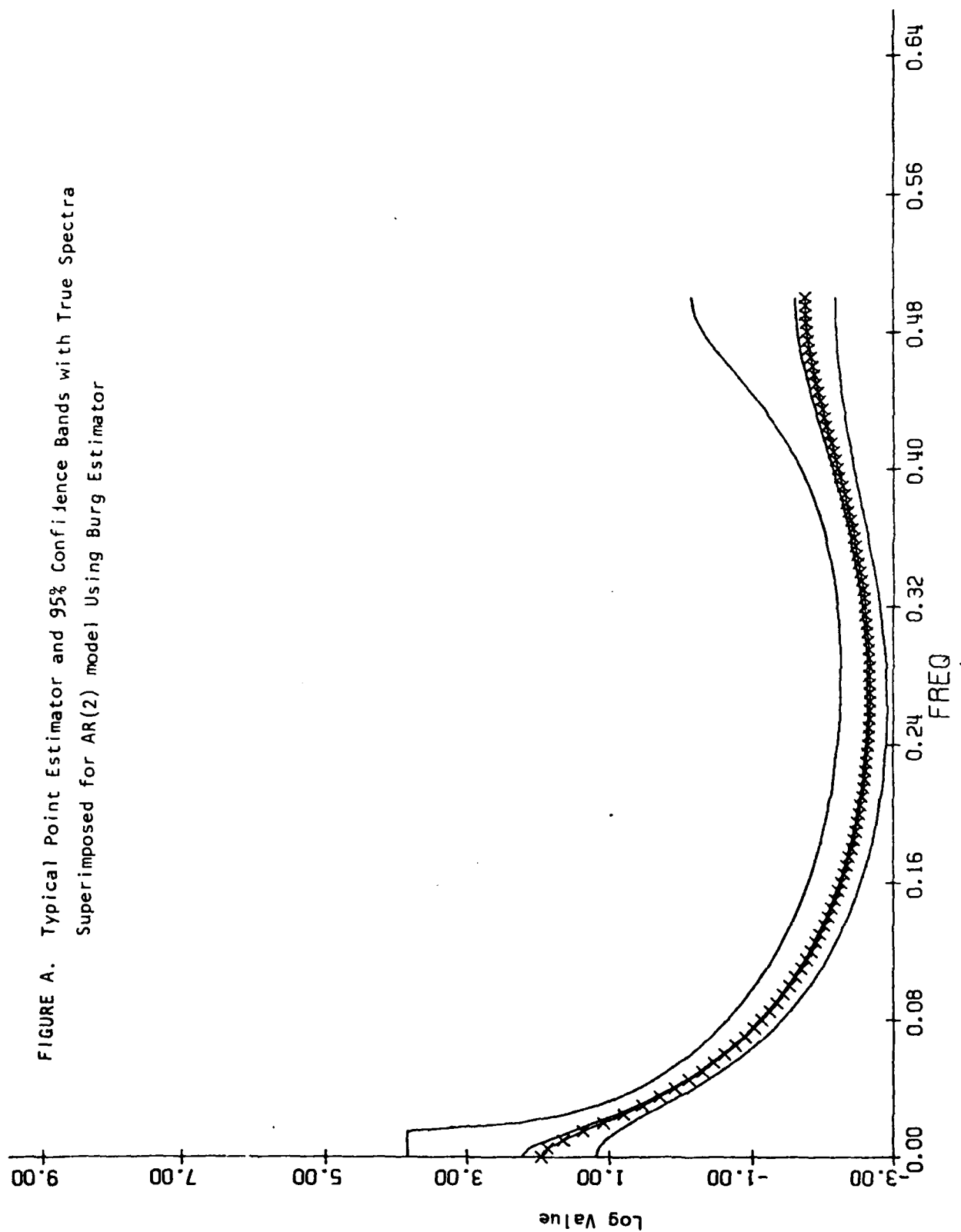
On the graphs, the largest noninfinite value of upper limits is assigned to $1/(\hat{h}(\omega) - S(\omega))$ whenever $\hat{h}(\omega) - S(\omega) < 0$. Examples of this include frequencies near .12 for the AR(4) model and those around .32 for the AR(5). We have found this to be a useful diagnostic in practice as peaks such as those in these simulated data may in fact be indicating the existence of deterministic components in real data in which case the spectral density does not exist.

In summary then, the confidence bands given in section 2 appear to work well on a wide variety of autoregressive models.

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and Autoregressive Decomposition," Rev. of Geophys. and Space
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FIGURE A. Typical Point Estimator and 95% Confidence Bands with True Spectra
Superimposed for AR(2) model Using Burg Estimator



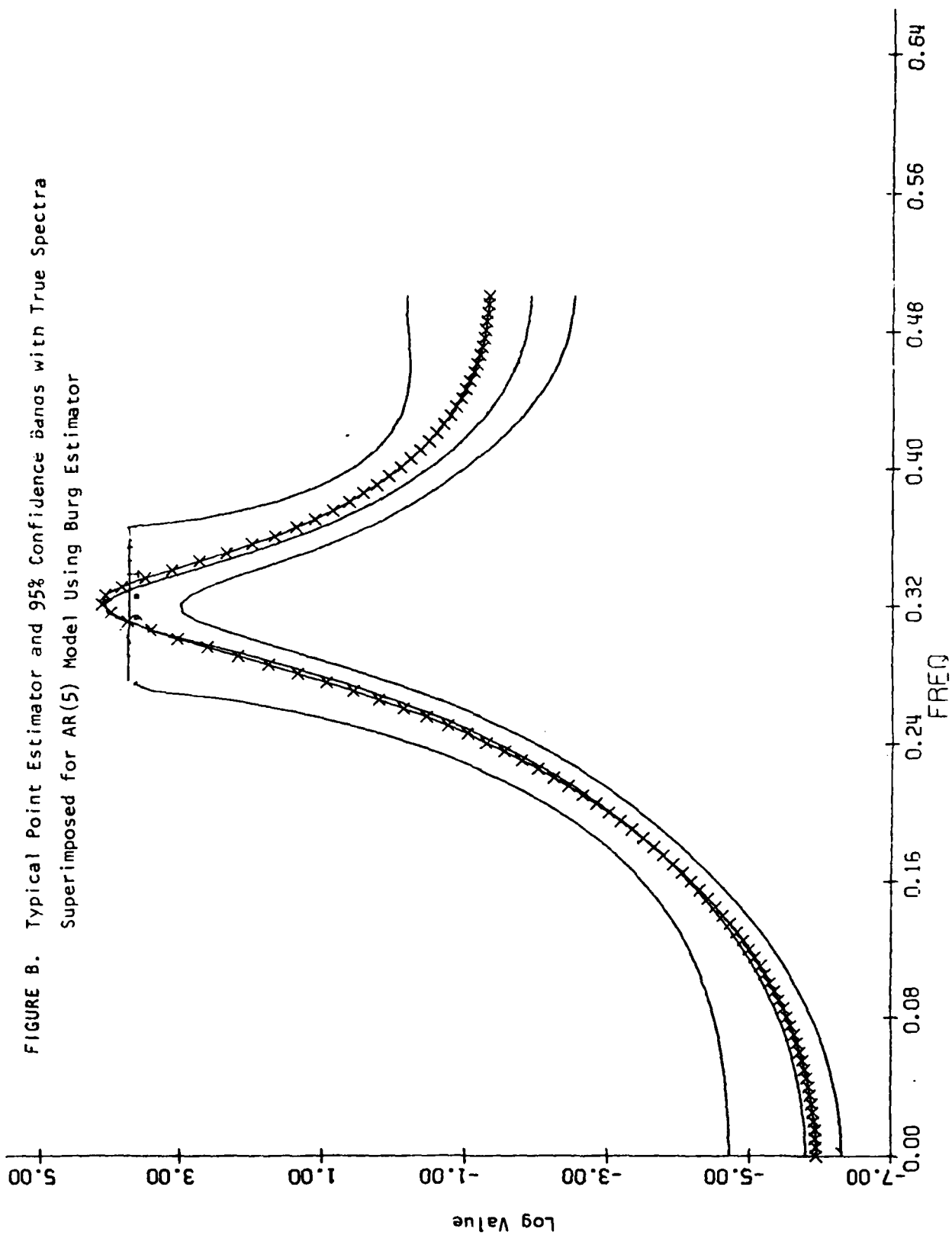
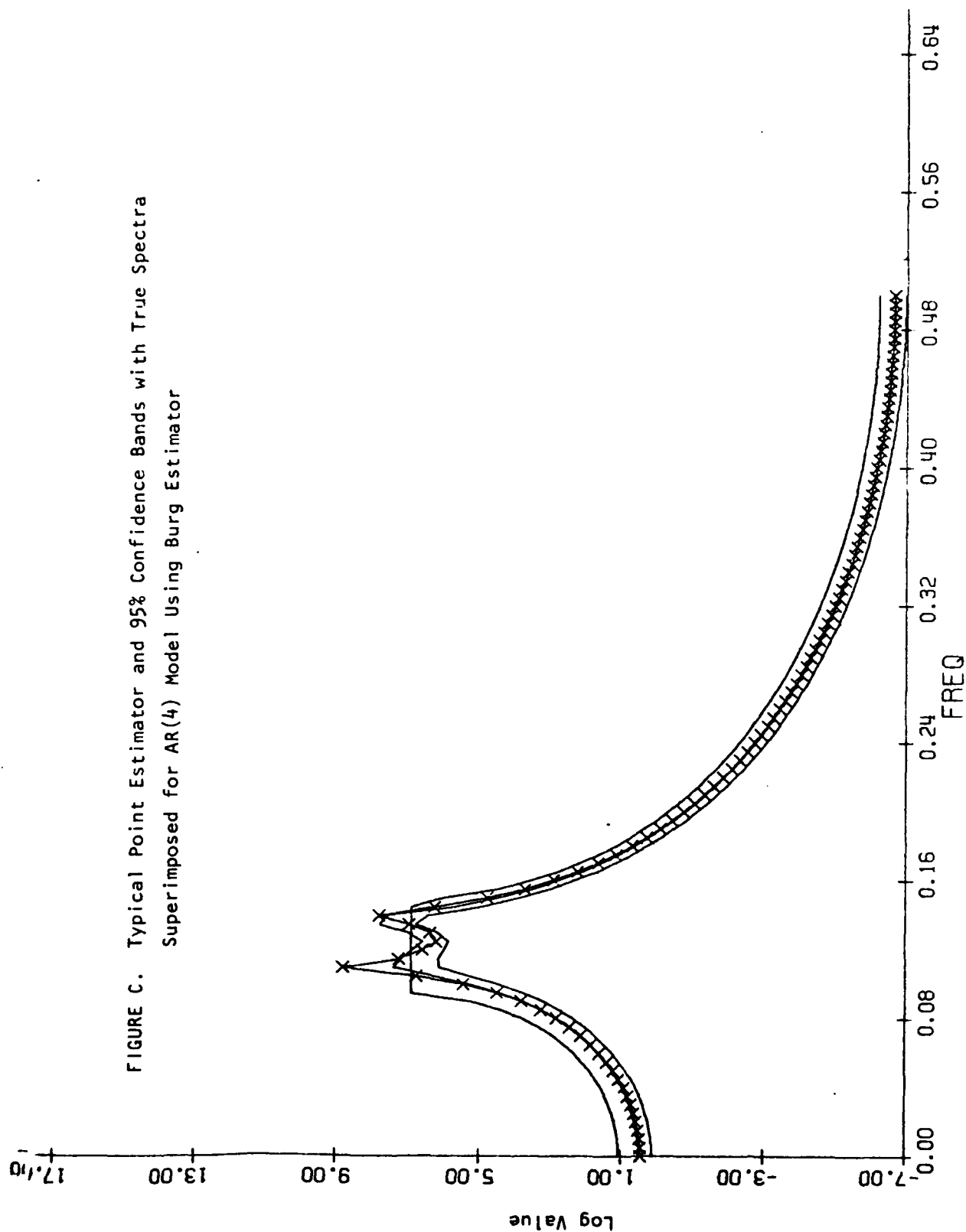


FIGURE C. Typical Point Estimator and 95% Confidence Bands with True Spectra
Superimposed for AR(4) Model Using Burg Estimator



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